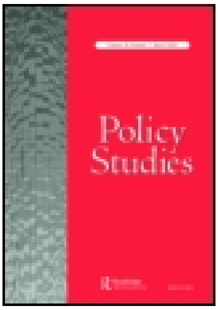

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Shock and awe: the effects of disinformation in military confrontation

Matthew T. Clements[a]

[a] Economics Department, St. Edward's University, Austin, TX, USA

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Shock and awe: the effects of disinformation in military confrontation

Matthew T. Clements*

Economics Department, St. Edward's University, Austin, TX, USA

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This paper analyzes the effects of disinformation in a military conflict. If one army distorts its opponents' perception of its ability, this will create a greater propensity for soldiers on the opposing side to surrender. The sender of disinformation will thus have a greater probability of victory. However, disinformation may also lengthen the battle and increase the total number of casualties. This depends not only on the degree of disinformation but also on whether and to what extent the sender of disinformation is superior to the receiver.

Keywords: asymmetric information; belief; updating; disinformation; warfare

1. Introduction

Prior to and during the war between the US and Iraq in 2003, a widely discussed component of US strategy was 'shock and awe.' Such a strategy comprises several elements, the combined aim of which is to achieve 'rapid dominance' over an enemy (Ullman and Wade 2003). In the popular understanding, effective use of shock and awe would result in the war essentially ending before it began: that the Iraqi army would be so intimidated by US forces that they would lay down their arms immediately. When this did not happen – when the war had not come to an end within a few days – a common view was that the shock and awe strategy had not been successful (see, for example, Kaplan 2003). This is, however, a narrow view of success. One could construe the strategy to be successful, at least to some degree, if it has the effect of decreasing the duration of the war or lessening the loss of life, or if it simply improves the probability of victory. At the same time, it is possible that these measures are at odds with each other, so that the strategy is not unambiguously positive. Consideration of all such factors is necessary before making any judgment of the desirability of shock and awe.

This paper considers the effects of one potential aspect of shock and awe: misleading the opposing side regarding one's military ability.[1] A key issue in analyzing the effects of such disinformation is how we model the behavior of the agents who receive the information. One view might assume that there is only one decision-maker on each side of the conflict: a general gives orders to soldiers, who must then follow them. An alternative view is to consider each individual soldier as a decision-maker: at each stage of the battle, each soldier decides whether or not to fight. In this view, soldiers are economic agents in the sense that they perceive a cost and a benefit of fighting (discussed in the following section). The purpose of this paper is to answer this question: Given that

*Email: mattcl@stedwards.edu

it is possible (although presumably costly) to send disinformation, what is the benefit of doing so? The model here is in a very reduced form and is intended to target this question. I do not parameterize the cost of disinformation, nor do I consider the mechanism by which false information is communicated. I simply assume that one side is able to manipulate the opponent's prior belief of its ability and examine the effects of this. The results of this paper inform the decision of whether and to what extent to disinform. Considering the cost and the mechanism of disinformation explicitly could present other interesting issues, but at its simplest would be a trivial exercise.

There is a substantial literature dealing with aggregation of public information (see, for example, Banerjee 1992; Bikhchandani, Hirshleifer, and Welch 1992; Gul and Lundholm 1995). In this literature, individual agents have private information about some common value. The informational structure in this paper is in a similar vein: individual soldiers on one side of a conflict have imperfect information about a common variable, the opposing side's ability. Inferences of the other side's ability affect individual soldiers' decisions of whether to fight or to surrender. There can be a cascading effect of surrendering – if some soldiers on one side surrender, the remaining soldiers are more vulnerable to the opposing side and are thus more likely to surrender – which is analogous to an information cascade. There is also a literature concerning the sending of false information (see, for example, Fudenberg and Tirole 1986), where the issue is what actions one player can take in order to induce an incorrect inference on the part of another player. Hendricks and McAfee (2006) consider the sending of false information within the context of war, but again the key issue is how to communicate this false information successfully. Zhuang and Bier (2011) and Zhuang, Bier, and Alagoz (2010) also endogenize the sender's decision of whether to deceive the receiver. In the three preceding references, there is a single decision-maker on each side, as opposed to a large number of receivers in the present paper. Costa and Kahn (2003) examine the factors that influence soldiers' propensity to surrender during the American Civil War. This could be viewed as an analysis of the parameters underlying the model presented below.

In the following section, I present a model of a military conflict in which soldiers make a fight-or-surrender decision at each stage. In Section 3, I present the implications of this model, some of which are subtle and nonintuitive. For example, when we view soldiers as individual decision-makers, disinformation always increases one side's probability of victory, but it can either increase or decrease the expected loss of life and duration of the conflict. The directions of these effects depend on the relative strength of the side that spreads false information and how strong the disinformation is. Section 4 concludes.

2. The model

Two armies face each other in battle. Over the course of the battle, each side decreases in size because of deaths and surrenders. The battle ends when there are essentially no soldiers left on one side who are willing to fight. The two opposing sides are indexed by i and j (or will alternatively be referred to as sides 1 and 2). One side begins with a mass of soldiers M_i, where the mass of an individual soldier approaches zero while the number of soldiers approaches infinity. Both masses are normalized to 1; the implications of armies of different sizes will be similar to those of different abilities, parameterized below. The mass remaining on one side at the beginning of any period t, after deaths and surrenders from previous periods, is M_{it} (I will suppress t when there is no ambiguity in doing so).

The *ability* of the soldiers on one side is $a_i > 0$ (where a_1 and a_2 may or may not be equal), and the *valor* of soldiers on each side is uniformly distributed[2] on $[0,\bar{v}]$, where the valor of soldier k on side i is v_{ik}. Valor encapsulates anything that motivates a soldier to fight. It could be that a soldier values victory and wants to contribute to the probability of victory, values fighting for its own sake, fears punishment for surrendering, etc. In any event, each soldier has some inclination to fight but would also like to preserve his own life. In each round of battle, a soldier fights if his valor is greater than his probability of death and surrenders otherwise.[3] Formally, a soldier who does not fight obtains a utility of 0. If soldier k fights, his expected utility is

$$v_{ik} - f(a_i, a_j, M_i/M_j, \beta), \tag{1}$$

where f is the probability of a soldier on side i being killed and $\beta > 0$ is a lethality parameter. As the technology of warfare improves, β increases (e.g., β will be higher if soldiers fight with guns rather than swords). I will use f_i to denote $f(a_i,\ a_j,\ M_i/M_j,\ \beta)$, and similarly for f_j, and assume the following properties:

$$\frac{\partial f_i}{\partial a_i} < 0, \frac{\partial f_i}{\partial M_i} < 0, \frac{\partial f_i}{\partial a_j} > 0, \frac{\partial f_i}{\partial M_j} > 0, \tag{2}$$

$$\frac{\partial^2 f_i}{\partial a_i^2} < 0, \frac{\partial^2 f_i}{\partial M_i^2} > 0, \frac{\partial^2 f_i}{\partial a_j^2} < 0, \frac{\partial^2 f_i}{\partial M_j^2} < 0, \frac{\partial^2 f_i}{\partial a_i \partial \beta} < 0. \tag{3}$$

The first-order derivatives are intuitive, and the second-order derivatives correspond to diminishing marginal effects of army size and ability. The condition on $\partial^2 f_i/\partial a_i \partial \beta$ means that the marginal effect of lethality diminishes as ability increases, and vice versa; i.e., improvements in the technology of warfare are more beneficial for low-ability armies.

'Surrendering' simply means that the soldier does not fight, whether it is because he surrenders to the other side or runs away. Once a soldier has surrendered, he may not re-enter the battle. An individual's net propensity to fight changes over the course of a battle as one side gains an advantage over the other. If many soldiers on side 1 are killed, the probability of death for the remaining soldiers on side 1 increases, and thus these soldiers are more likely to surrender. This effect would go in the opposite direction for side 2. Valor is constant over the course of a battle, but the results here are consistent with a model in which valor decreases as the probability of death increases. If, for example, there is less stigma associated with surrender when the probability of death is very high, this is operationally equivalent to a lower valor. Since soldiers are heterogeneous with respect to valor, it is possible for some fraction of soldiers to surrender in any given period, not just all or none.[4]

The battle proceeds in discrete stages. In each stage t, all soldiers on the battlefield fight. Some of them die: D_{1t} and D_{2t}. Of the remainder, some surrender: S_{1t} and S_{2t}. The remaining mass on each side goes on to the next stage. The battle ends when the mass on one side is less than ε, where $\varepsilon > 0$. Note that it is not generally possible for either side to be completely eliminated: neither side can be killed off because the probability of death will generally be strictly less than one, and neither side will surrender completely because some fraction of soldiers will have a valor arbitrarily close to 1. The stopping rule ensures that every battle will end and does not otherwise distort the results as long as ε is sufficiently small.[5]

Soldiers on side j have a prior belief of the ability of soldiers on side i, $\hat{a}_i$. The prior may be incorrect, and furthermore, side i may be able to manipulate $\hat{a}_i$, perhaps through a conspicuous display of might. I define *disinformation* to be the difference between the sender's true ability and the receiver's prior:

$$d_i \equiv \hat{a}_i - a_i, \tag{4}$$

where $d_i \geq 0$. As noted above, the informational structure is reduced-form; I do not model the process through which soldiers' prior beliefs change. A key presumption of this paper is that it is possible to deceive soldiers about the ability of the enemy they face in battle. Two classic treatises of military strategy support this: in *The Art of War*, Sun Tzu (1963) states that 'all warfare is based on deception,' and in *On War*, Carl von Clausewitz (1968) refers to the 'fog, friction, and fear' that influences soldiers in the heat of battle and can be manipulated by the opposition. Given the possibility of deception, I will examine the effects of the deception on the outcome of the battle, not only in terms of the probability of victory but also in terms of duration and casualties. There may be costs or other factors limiting d_i; I simply take the extent of disinformation to be exogenous. The results of this model could be embedded in a larger model that considers cost explicitly.

The prior on the opposition's ability is updated over the course of battle, as soldiers observe the number of deaths on each side. Deaths are observed imperfectly, and thus an incorrect prior may persist for some time. Formally, the number of deaths observed by the soldiers on one side is the actual number of deaths plus a noise parameter, where the noise parameter has mean 0 and standard deviation σ. If $\sigma = 0$, soldiers update their prior to the correct value after one round of battle. If $\sigma > 0$, soldiers' beliefs will, on average, converge to the correct value over the course of the battle as soldiers update in Bayesian fashion, using their observation of the battle's history.[6] Note that, even if the prior is correct, soldiers will update it over the course of battle. In each round of the battle, soldiers fight, and those who are not killed update their belief of the other side's ability based on their (imperfect) observation of the number killed. Soldiers then decide whether to surrender based on this updated belief. Soldiers who do not surrender advance to the next round. At any stage, side j's updated belief of side i's ability is a_{it}. All parameters except a_i and a_j are common knowledge.

3. Results

All proofs are in Appendix 1. Without loss of generality, I assume that $a_1 \geq a_2$; i.e., if one side is superior, it is side 1.

3.1. Correct prior beliefs

I first consider the case in which each side has the correct prior belief about the other side's ability. To make the implications of the model very clear, I also consider the special case in which $\sigma = 0$, i.e., soldiers observe deaths perfectly throughout the battle. There is then no way for the prior to be distorted because of random noise. When armies are evenly matched ($a_1 = a_2$), the remaining mass of soldiers on each side will be equal at any time, because at any time the probability of death is the same for a soldier on either side. In fact, the probability of death remains constant throughout the battle, because the relative proportions of soldiers on each side are constant. Because of this, there can only be surrenders in the first round of battle: those whose valor is less than the probability of death surrender at the first opportunity, and those who remain never surrender because the probability of death never increases. The armies fight until the mass on both sides drops below ε at the same time. If $a_1 > a_2$, then there is an accompanying difference in the probability of death: $f_1 < f_2$. There are deaths on both sides in the first round, but there are fewer on side 1; and in addition, a greater mass of soldiers on side 2 surrenders. Thus the difference in probabilities becomes more pronounced in the next period. There will be no more surrenders on side 1 because f_1 can only decrease, but there will continue to be surrenders on side 2 as f_2 increases. The difference in probabilities increases as the battle continues, and side 1 wins for certain because side 2's mass decreases at a greater rate.

When $\sigma > 0$, if one army is superior, it is more likely to win. However, it is possible that the inferior army wins, if noisy observation of deaths distorts the superior (inferior) side's prior in such a way that there are more (fewer) surrenders from the superior (inferior) side than expected. In this case, the mass of the inferior army relative to that of the superior army may be large enough to overcompensate for the difference in abilities. It is generally the case that one side's probability of victory is increasing in its own ability, and if there is a sufficient difference in abilities, the battle ends immediately because a sufficiently large mass of soldiers in the inferior army surrenders in the first round. Generally, the more closely matched the two sides are, the longer the duration of the battle and the greater the number of deaths. Closely matched armies gradually chip away at each other until one side's mass eventually falls below ε. The greater the difference in the armies' abilities, the more surrenders there will be on the inferior side at the outset. The greater number of surrenders leads to more surrenders in successive periods because the inferior side will be smaller and the probability of death will be higher for those remaining. The disparity in abilities thus causes the battle to end sooner, and it may decrease the total casualties as well.

These results are summarized in the following proposition:

Proposition 1

If $a_1 = a_2$, then each side has an equal probability of victory. If $a_1 > a_2$, then side 1 is more likely to win, and side 1 will win for certain if $\sigma = 0$. The probability of victory for side 1 is nondecreasing in a_1 and nonincreasing in a_2. The expected duration is nonincreasing in a_1 and nondecreasing in a_2.

3.2. Distorted prior beliefs

Now I assume that one side is able to distort its opponent's prior belief of its ability. I assume that disinformation is always successful in that it does induce an incorrect belief, at least temporarily. There could be costs or constraints associated with disinformation that are not modeled here. Some results here are very intuitive. When $\sigma = 0$, the only effect disinformation can have is to increase surrenders in the first round, before any deaths are observed and the prior is updated to the correct value, which may be sufficient for the inferior army to win. Generally, disinformation increases an army's probability of victory. Any army sending a sufficient degree of disinformation is certain to win. If one side is very inferior, the only way it can win is by distorting the prior so much that all of the soldiers on the other side surrender as soon as possible, or by inducing enough surrenders that the inferior army has an advantage by virtue of its relative size. Otherwise, the sender obtains an initial advantage – more soldiers on the opposing side surrender than would do so if they had the correct prior – which is eroded over the course of battle. If the initial advantage is large enough and persists long enough, the inferior side can win.

Proposition 2

The probability of victory for side i is nondecreasing in d_i. For d_i sufficiently high, side i is certain to win.

The remaining results are delineated by whether the inferior or the superior army is the sender of disinformation. If the superior army is the sender (or if sender and receiver are evenly matched), then disinformation is expected to have positive results in every sense: the sender's probability of victory increases, the expected number of deaths decreases, and the expected duration of the battle decreases. In the extreme, disinformation induces immediate surrender of all of the soldiers on the inferior side. In less extreme cases, there are initially more surrenders on the inferior side than there would be under the correct prior belief. Because of this, in successive rounds, the inferior side is smaller, leading to more surrenders on the inferior side and fewer on the superior side. Those soldiers on the inferior side that decide to fight are more likely to be killed, but the effect on surrenders dominates. Intuitively, the reason for this is that the increase in the probability of death also increases the individual soldier's propensity to surrender, and there is always an opportunity to surrender before being exposed to the possibility of death. Since the disinformation increases attrition of the inferior side, the duration of the battle is shorter. Soldiers on the superior side are more likely to fight but only because they are less likely to be killed. The only way that more on the superior side would die would be if the duration of the battle were longer.

Proposition 3

The expected number of deaths on each side is nonincreasing in d_1; the expected number of surrenders on side 1 (side 2) is nonincreasing (nondecreasing) in d_1; and the expected duration is nonincreasing in d_1.

The results are rather different if the sender is the inferior army. As above, disinformation increases the number of surrenders on the receiver's side and decreases surrenders on the sender's side. However, the duration of the battle and the number of deaths on each side are increasing in disinformation when disinformation is small and decreasing when disinformation is large. For a sufficient degree of disinformation, everyone on the superior (receiving) side surrenders in the first period. When disinformation is slightly less extreme, the sender has a large initial advantage because many on the receiving side surrender. The battle ends quickly and with few deaths. On the other hand, if the disinformation is slight, the disinformation leads to fewer surrenders on the inferior side and more on the superior side, which serves to lengthen the battle and expose more soldiers to the probability of death. It is as if the small amount of disinformation adds a preliminary phase to the main battle, where the main battle begins when the advantage of disinformation has dissipated. The preliminary phase extends the duration and increases the number of deaths. For more severe disinformation, on the other hand, the inferior side's informational advantage never dissipates entirely; it is as if the battle ends during the preliminary phase.

Proposition 4

Assume $a_1 > a_2$. The expected number of surrenders on side 2 (side 1) is nonincreasing (nondecreasing) in d_2. There exists d^* such that:

(1) For $d_2 < d^*$, the expected number of deaths on each side is nondecreasing in d_2 and the expected duration is nondecreasing in d_2.
(2) For $d_2 > d^*$, the expected number of deaths on each side is nonincreasing in d_2 and the expected duration is nonincreasing in d_2.

The effects of lethality also become complicated if the inferior army is the sender of disinformation. Increasing lethality always causes more surrenders on the superior side and fewer deaths on the inferior side, by the same reasoning behind Proposition 1. If lethality is low, then increasing lethality increases the number of surrenders on the inferior side but increases the number of deaths on the superior side, i.e., the effect on inferior surrenders is the same as in the case of no disinformation, but the effect on superior deaths is reversed. This is because for small increases in β in this region, the number of superior surrenders increases slightly but the number of inferior surrenders increases more dramatically. If lethality is high, then increasing lethality decreases the number of surrenders on the inferior side and decreases the number of deaths on the superior side; the effect on inferior surrenders is reversed, and the effect on superior deaths is the same as in the case of no disinformation. This is because for small increases in β in this region, the number of superior surrenders increases substantially, which makes soldiers on the inferior side more willing to fight.

Proposition 5

If $a_1 > a_2$, $d_2 > 0$, and $d_1 = 0$, the expected number of deaths on side 2 is nonincreasing in β and the expected number of surrenders on side 1 is nondecreasing in β.

Proposition 6

If $a_1 > a_2$, $d_2 > 0$, and $d_1 = 0$, then there exists β^* such that the following hold.

(1) For $\beta < \beta^*$:
- the expected number of surrenders on side 2 is nondecreasing in β.
- the expected number of deaths on side 1 is nondecreasing in β.

(2) For $\beta > \beta^*$:
- the expected number of surrenders on side 2 is nonincreasing in β.
- the expected number of deaths on side 1 is nonincreasing in β.

These results depend on the assumption that $\frac{\partial^2 f_i}{\partial a_i \partial \beta} > 0$. When lethality magnifies the effect of ability, it also magnifies the effect of disinformation. If lethality is high, increases in lethality cause relatively more soldiers on the superior side to surrender because they mistakenly think that their probability of death has increased substantially.

4. Conclusion

I have examined the effects of sending disinformation in a military conflict as part of a broader strategy of shock and awe. The potential success of such a strategy can be measured in several ways: increase in the probability of victory, decrease in the loss of life, or decrease in the duration of the conflict. Disinformation can have unambiguously positive effects, but it can also have mixed effects, such as increased probability of victory as well as increased loss of life on both sides. Thus, even if we neglect the cost of spreading disinformation, it may not be worthwhile to pursue. An inferior army that is able to create only minor disinformation will increase its probability of victory but can also expect to incur more casualties. For any army, the cost of actually creating the disinformation (which has not been modeled here) would be weighed against the benefit, in terms of potentially reducing the loss of life and other costs of engaging in warfare. A natural extension would be to evaluate this tradeoff, which might include endogenizing the extent of disinformation itself. In any case, a decision of whether to employ a shock and awe strategy requires an understanding of all of the effects of the strategy. The contribution of this paper is to illuminate those effects.

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Notes

1. The general aim of shock and awe, according to Ullman and Wade (2003), is 'to affect the will, perception, and understanding of the adversary to fight' (xxiv). There are a number of potential facets of such a strategy, any subset of which might be employed. To what extent the American military strategy in Iraq depended on disinformation is debatable. The purpose of this paper is to consider the effects of disinformation in whatever context it might be used, not to provide a comprehensive or definitive analysis of the conflict in Iraq.
2. The results hold as long as there is not too much mass in the tails of the valor distribution.
3. It is possible for any given soldier's valor to equal 1, and this can be taken to mean that the soldier will fight even in the face of certain death. In terms of the model, the soldier would technically be indifferent between fighting and surrendering, but the resolution of this indifference does not affect the results.
4. There need only be one source of heterogeneity in the model to allow this; heterogeneity of a_i complicates the model substantially but does not add any insight. Similar complications arise if the two sides have entirely different distributions of valor, but again without significant gain in insight.
5. In the proofs of the results, I will assume that the probability of one side's winning includes the probability of a draw (where both sides' masses fall below ε during the same period). In most cases, the probability of a draw is arbitrarily small.
6. This is the simplest possible informational structure: all soldiers are unaware of the extent of disinformation and update beliefs in a limited way. If soldiers' beliefs were more complex – e.g., if they were able to draw inferences about disinformation, or about their own propensity to die in future periods – the qualitative changes in the results below would generally be predictable. In particular, disinformation would be less effective in the presence of more sophisticated beliefs.

Notes on contributor

Matthew Clements is an Associate Professor of Economics at St. Edward's University. He blogs at http://www.phlegmaticblog.com/.

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Appendix 1

First I expand upon the notation in the text. The following will be used in all of the subsequent proofs.

The mass of soldiers on side i at the start of period t is M_{it}; the mass at the end of period t or beginning of period t+ 1 is $M_{i,t+1}$. Deaths, D_{it}, occur first each period, and then surrenders, S_{it}. The following relations follow from the properties discussed above:

$$D_{it} = M_{it} f_{it} \tag{5}$$

$$S_{it} = M_{it}(1 - f_{it}) \left[\frac{\hat{f}_{it} - v_{it}}{\bar{v} - v_{it}} \right] \tag{6}$$

$$M_{i,t+1} = M_{it}(1 - f_{it}) \left[\frac{\bar{v} - \hat{f}_{it}}{\bar{v} - v_{it}} \right] \tag{7}$$

where $\hat{f}_{it} = f(a_i, \hat{a}_{jt}, M_{it}/M_{jt}, \beta)$ and v_{it} is the minimum valor remaining on side i (i.e., everyone for whom $v_{ik} < v_{it}$ has already surrendered or been killed). The state of the battle at any time t is defined by the remaining masses and beliefs of the soldiers on each side, $\Phi_t = (M_{1t}, M_{2t}, \hat{a}_{1t}, \hat{a}_{2t})$. The probability side i wins in period t is the probability that $M_{i,t+1} > \varepsilon$ and $M_{j,t+1} < \varepsilon$, and is denoted W_{it}. (The probability of a draw, wherein $M_{i,t+1} < \varepsilon$ and $M_{j,t+1} < \varepsilon$, is negligible for small ε but is significant for large β.) The probability that side i wins at some point in the battle is then $W_i = \sum_{t=1}^{\infty} W_{it}$. The expected duration of the battle is $T = \sum_{t=1}^{\infty} t(W_{it} + W_{jt})$. Let T_t denote the expected remaining duration at time t: $T_t = \sum_{s=t}^{\infty} s(W_{is} + W_{js})$. For the results regarding duration, the following lemma, stating that the remainder of the battle is expected to be longer if the armies are more evenly matched, will be useful.

Lemma 1

T_t is decreasing in $|M_{it} - M_{jt}|$.

Proof

It is sufficient to show that $\sum_{t=0}^{n}(W_{it} + W_{jt})$ is increasing in $|M_{it}$-$M_{jt}|$ for every n; i.e., when there is a larger difference in the two masses, probability mass shifts inward from later periods to earlier periods. WLOG assume $M_{it} \geq M_{jt}$. Consider the effect of an increase in $(M_{it} - M_{jt})$ in period t. Clearly, W_{it} increases and W_{jt} decreases. Furthermore, the increase in W_{it} dominates the decrease in W_{jt} so that $(W_{it} + W_{jt})$ increases. To see this, note that the change in $(M_{it} - M_{jt})$ increases D_{jt} and decreases D_{it} for certain. There is also an increase in $E(S_{jt})$ and a decrease in $E(S_{it})$, both directly from the change in $(M_{it} - M_{jt})$ and indirectly from the changes in D_{it} and D_{jt}. Consider two components of the probability of victory, arising from deaths or from surrenders on the opposing side. Each component increases for side i, but only the surrender component decreases for side j since deaths are deterministic. Furthermore, given the properties of f, the decrease in the surrender component of victory for side j must be smaller than the increase in the surrender component of victory for side i. Thus $(W_{it} + W_{jt})$ increases. Moreover, masses are more skewed in the following period, and the effects on the probabilities of victory are more pronounced. So for all t, $(W_{it} + W_{jt})$ increases, and this is conditional on the battle continuing until period t. Thus $\sum_{t=0}^{n}(W_{it} + W_{jt})$ is increasing.

Proof of Proposition 1

The result that the probability of victory is equal for both sides when $a_1 = a_2$ is trivially true: the expressions for W_{1t} and W_{2t} are identical for every t, and so $W_1 = W_2$. Now consider the effect of either increasing a_1 or decreasing a_2. Because deaths are deterministic, there will be a decrease in D_{11} and an increase in D_{21}. Since the only stochastic element of surrenders is $\hat{f}$, expected surrenders will be

$$E(S_{it}) = M_{it}(1 - f_{it})\left[\frac{E(\hat{f}_{it} - v_{it})}{\bar{v} - v_{it}}\right]. \tag{8}$$

Both of these effects tip the probability of victory in side 1's favor: W_{11} increases and W_{21} decreases. Now consider an arbitrary period t, and the effect of changes in a_1 or a_2 on $E(W_{i,t+1}|\Phi_t)$. Because of soldiers' stochastic observation of deaths, it could be that $M_{1t} < M_{2t}$ and $\hat{a}_{1t} < \hat{a}_{2t}$, even though $a_1 > a_2$. However, conditional on Φt, an increase in a_1 or decrease in a_2 has the effect of decreasing $D_{1,t+1}$ and $E(S_{1,t+1})$ and increasing $D_{2,t+1}$ and $E(S_{2,t+1})$. Since this is true for every t, we can sum over the conditional expected values to obtain the result for W_1 and W_2. When $\sigma = 0$, surrenders as well as deaths are deterministic, and $W_1 = 1$ trivially. The result regarding duration follows directly from the lemma: an increase in (a_1-a_2) increases (M_1-M_2), which leads to a shorter expected duration.

Proof of Proposition 2

The effects of changes in d_i are similar to the effects of changes in a_i, except that d_i only affects surrenders and does not affect deaths directly. For d_i sufficiently high, all soldiers on side j surrender immediately. For lower d_i, the proof proceeds as in the previous proposition. The increase in d_i increases W_{i1} and decreases W_{j1} because there are more surrenders on side j. For an arbitrary period t, conditional on Φ_t, an increase in d_i decreases $E(S_{i,t+1})$ and increases $E(S_{j,t+1})$. Summing over the conditional expected values, W_i must increase.

Proof of Proposition 3

If the distribution of valor on side 2 is sufficiently skewed toward the high end, then no one on side 2 ever surrenders, and d_2 has no effect at all. When this is not the case, first note that

$$E(D_i) = E\sum D_{it} = \sum E(D_{it}) = \sum\sum E(D_{it}|\Phi_t)Pr(\Phi_t),$$

with a similar expression for $E(S_i)$. In the same vein as the previous proofs, consider the effects of d_1 in period 1. There is an increase in S_{21} and thus a decrease in M_{22}. This increases f_{22} and $\hat{f}_{22}$ and decreases f_{12}. In period 2, there is a higher probability of death on side 2 but also a smaller mass. It is straightforward to show that $M_{22}f_{22}$ decreases. Intuitively, it is the change in mass that increases f_2, but $\hat{f}_2 > f_2$, and thus the effect on surrenders is larger. Similar reasoning applies for an arbitrary period t, conditional on Φ_t, and summing over the conditional expected values yields the results for deaths and for surrenders. As in Proposition 1, the result regarding duration follows from the lemma.

Proof of Proposition 4

For d_2 sufficiently large, all soldiers on side 1 surrender at the first opportunity, and the number of deaths is minimized. For some smaller d_2, the battle continues into the second period, with $M_1 << M_2$. Using the lemma above, the expected duration of the battle is shorter, the larger is d_2. Clearly, the number of deaths is also smaller, the larger is d_2. On the other hand, for d_2 small, expected surrenders on side 1 are slightly higher than when $d_2 = 0$. Then in the second period, masses are less skewed than they would otherwise be, implying a longer expected duration. A larger number of deaths follow from the longer duration. The existence of d^* follows from the continuity of f.

Proofs of Propositions 5 and 6

These proofs are very similar to the proofs for Propositions 3 and 4. Mathematically, changes in β are analogous to changes in d.